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An Extension of Strong Uniqueness to Rational Approximation

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In this paper the concept of strong uniqueness is extended to non-normal rational minimization problems. A characterization of those problems which have strongly unique solutions is given. To obtain this characterization a refinement of the Kolmogorov criterion is proved.

1. INTRODUCTION

Let S be a compact Hausdorff space, $S \neq \varnothing$, and define the compact Hausdorff space $T := \{-1, 1\} \times S$. Let $B, C: S \to \mathbb{R}^N$ be continuous functions such that the set

$$U := \bigcap_{s \in S} \{v \in \mathbb{R}^N \mid \langle C(s), v \rangle > 0\}$$

is non-empty. Let $\gamma: T \to \mathbb{R}$ be continuous non-negative and for $(v, z) \in U \times \mathbb{R}$ define $p(v, z) := z$.

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For each $x \in C(S)$ consider the minimization problem $MPR(x)$.

Minimize $p(v, z)$
subject to

$$\underset{(\eta,s) \in T}{\forall} \eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s) z \leqslant \eta x(s).$$

A particular case is given by the following.
Let $g_1, g_2, ..., g_l, h_1, h_2, ..., h_m \in C(S)$ be such that

$$\left\{\beta \in \mathbb{R}^m \,\middle|\, \underset{s \in S}{\forall} \sum_{i=1}^{m} \beta_i h_i(s) > 0\right\}$$

is non-empty and define $N := l + m$,

$$B(s) := (g_1(s), g_2(s), ..., g_l(s), 0, 0, ..., 0),$$
$$C(s) := (0, 0, ..., 0, h_1(s), h_2(s), ..., h_m(s)).$$

As was shown in [3], this particular case contains certain classes of rational Chebyshev approximation problems, f.e. weighted, one-sided and unsymmetric problems.

Define the set

$$V := \left\{\frac{\langle B, v\rangle}{\langle C, v\rangle} \in C(S) \,\middle|\, v \in U\right\}.$$

A pair $(\langle B, v_0\rangle / \langle C, v_0\rangle, z_0) \in V \times \mathbb{R}$ is also called a solution of MPR(x), whenever (v_0, z_0) is a solution of MPR(x). For each $r_0 \in V$ we define the linear subspace

$$H_0 := \left\{v \in \mathbb{R}^N \,\middle|\, \underset{s \in S}{\forall} \langle r_0(s)\, C(s) - B(s), v\rangle = 0\right\},$$

and for each $v \in \mathbb{R}^N$ let φ_v be the angle between v and H_0.

For each $x \in C(S)$ we introduce the sets

$$Z_x := \left\{(v, z) \in U \times \mathbb{R} \,\middle|\, \underset{(\eta,s) \in T}{\forall} \eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s) z \leqslant \eta x(s)\right\}$$

and

$$V_x := \left\{\left(\frac{\langle B, v\rangle}{\langle C, v\rangle}, z\right) \in V \times \mathbb{R} \,\middle|\, (v, z) \in Z_x\right\}.$$

We denote by L the set

$$\{x \in C(S) \mid \mathrm{MPR}(x) \text{ has a solution}\}.$$

A solution (r_0, z_0) of the minimization problem $\mathrm{MPR}(x)$ is called strongly unique if and only if there exists a constant $K_1 := K_1(x) > 0$ such that

$$\forall_{(v,z) \in Z_x} \quad z - z_0 \geqslant K_1 \varphi_v. \tag{*}$$

In this paper we characterize those functions x in L such that $\mathrm{MPR}(x)$ has a strongly unique solution (r_0, z_0). It turns out that the Haar-condition in a certain finite subset of S is always sufficient for strong uniqueness and also necessary provided $\gamma(\eta, s) > 0$ for $(\eta, s) \in T$. We remark that these results are valid without assuming normality of the function x.

In the normal case (compare Section 5) we prove that condition (*) is equivalent to the usual definition of strong unicity, i.e.,

$$\forall_{(r,z) \in V_x} \quad z - z_0 \geqslant K_2 \|r - r_0\|_\infty \tag{**}$$

where $K_2 := K_2(x) > 0$. It is known that in the non-normal case even with Haar-condition in S the inequality (**) is not valid. Thus definition (*) of strong uniqueness extends the usual one in a natural way.

For rational Chebyshev approximation Cheney and Loeb [5] proved a strong uniqueness result of the type

$$\|x - r\|_\infty - \|x - r_0\|_\infty \geqslant K_3 \varphi_v^2 \tag{***}$$

assuming that x is normal and the Haar-condition is satisfied in S. This result was later extended by Brosowski [1] to the non-normal case. In view of Theorem 5.2 and Example 6.2 it is not possible to derive the strong uniqueness result (**) from (***). A direct proof of (**) was given by Cheney [4] assuming the Haar-condition in S. Later Loeb [8] estimated in the non-normal case the difference

$$\|x - r\|_\infty - \|x - r_0\|_\infty$$

essentially by $K_4 \cdot \varphi_0$ also assuming the Haar-condition in S.

In the proof of the sufficiency part of the strong uniqueness Theorem 4.1 we use a refinement of the Kolmogorov criterion, which in proved in Section 3. This refinement extends a result of Brosowski [2] in the linear case, who also used it to characterize functions with strongly unique best approximations.

Since the Haar-condition in S implies, of course, the Haar-condition in any finite subset of S, the various results mentioned above follow from our results. Also results of Loeb and Moursund [9] and of Taylor [10] for the case of one-sided rational Chebyshev approximation are included. In Theorems 4.2 and 5.2 we have strong uniqueness results in the parameter space which contain results of Cheney and Loeb [6] and Hettich and Zencke [7].

If condition (*) is satisfied for $\mathrm{MPR}(x)$ then we can derive in the case

$$T_c := \{(\eta, s) \in T \mid \gamma(\eta, s) > 0\}$$

compact a continuity result for the angle φ_v, i.e., there exists a constant $K_5 := K_5(x) > 0$ such that

$$\varphi_v \leqslant K_5 \|y - x\|$$

for all y in L, where v defines a solution of $\mathrm{MPR}(y)$. If x is a normal point, then we can derive from (**) a continuity result for the metric projection. We remark that in the case of usual Chebyshev approximation and in the case of one-sided approximation the set T_c is always compact.

We introduce some definitions and notations. For each $r_0 \in V$ define the linear space

$$\mathscr{L}(r_0) := \{\langle r_0 C - B, v \rangle \in C(S) \mid v \in \mathbb{R}^N\}.$$

Choose a basis $\varphi_1, \varphi_2, ..., \varphi_d$ of $\mathscr{L}(r_0)$ and define for each $t = (\eta, s)$ in T the vectors

$$G(t) := G(\eta, s) := \eta(\varphi_1(s), \varphi_2(s), ..., \varphi_d(s)).$$

A subset $M \subset T$ is said to be critical (with respect to r_0 in V) iff

$$0 \in \mathrm{con}(\{G(t) \in \mathbb{R}^d \mid t \in M\}).$$

For each $(r_0, z_0) \in V \times \mathbb{R}$, $z_0 > 0$, define

$$M_0 := \{(\eta, s) \in T \mid \eta(r_0(s) - x(s)) = \gamma(\eta, s) z_0\}.$$

A signature on S is a continuous mapping defined on a closed subset of S into $\{-1, 1\}$. In the following we assume that $x \notin V$ and that

$$\underset{s \in S}{\forall}\ \gamma(-1, s) + \gamma(1, s) > 0.$$

We define a signature ε_0 by setting $\varepsilon_0(s) = \eta$ for each $(\eta, s) \in M_0$. A signature ε is said to be critical iff

$$\{(\varepsilon(s), s) \in T \mid s \in \mathrm{DOM}(\varepsilon)\}$$

is a critical subset of T. A critical signature is called primitive, if it does not contain properly any other critical signature. We denote by Λ_0 the set of all primitive critical signatures contained in ε_0.

For each signature ε define the linear space

$$V(\varepsilon) := \left\{ v \in \mathbb{R}^N \;\middle|\; \underset{s \in \mathrm{DOM}(\varepsilon)}{\forall} \langle r_0(s)\, C(s) - B(s), v \rangle = 0 \right\},$$

and for each $v \in \mathbb{R}^N$ let $\varphi_v(\varepsilon)$ denote the angle between v and $V(\varepsilon)$. Further define

$$\Gamma_0 := \{(\varepsilon(s), s) \in M_0 \mid \varepsilon \in \Lambda_0\}$$

and

$$S_0 := \{s \in S \mid (\varepsilon_0(s), s) \in \Gamma_0\}.$$

Using Theorem 1.3 and Lemma 4.2 of [3] we have

THEOREM 1.1. *If (r_0, z_0) is a solution of $MPR(x)$, then ε_0 is a critical signature.*

This theorem implies that the sets Λ_0, Γ_0, and S_0 are non-empty provided (r_0, z_0) is a solution of MPR(x). In this case we denote the restriction of ε_0 to S_0 by $\tilde{\varepsilon}_0$.

2. A Lemma

LEMMA 2.1. *Let A be a non-empty bounded subset of $\mathbb{R}^N$ such that*

$$\underset{v \in H \setminus \{0\}}{\forall} \inf_{w \in A} \langle v, w \rangle < 0,$$

where $H := \mathrm{span}(A)$.

Then there exists a constant $K > 0$ such that

$$\underset{v \in \mathbb{R}^N}{\forall} \inf_{w \in A} \langle v, w \rangle \leqslant -K \|v\| \psi_v,$$

where ψ_v denotes the angle between v and $H^{\perp}$.

Proof. By hypothesis, we have

$$\underset{\substack{v \in H \\ \|v\| = 1}}{\forall} \Psi(v) := \inf_{w \in A} \langle v, w \rangle < 0.$$

Hence there exists $\alpha > 0$ such that

$$\Psi(v) \leqslant -\alpha$$

for each $v \in H$ with $\|v\| = 1$. If not there exists a sequence (v_n) contained in H such that $\|v_n\| = 1$, $\Psi(v_n) \to 0$, and $v_n \to v_0$. Since $\Psi(v_0) < 0$ there exists $w_0 \in A$ such that $\langle v_0, w_0 \rangle < 0$. Consequently,

$$\langle v_0, w_0 \rangle < \Psi(v_n) \leqslant \langle v_n, w_0 \rangle$$

for n large enough. For $n \to \infty$ we obtain

$$\langle v_0, w_0 \rangle < 0 \leqslant \langle v_0, w_0 \rangle,$$

which is a contradiction. By homogeneity, we have

$$\underset{v \in H}{\forall} \inf_{w \in A} \langle v, w \rangle \leqslant -\alpha \|v\|.$$

Now consider $v \in \mathbb{R}^N$ and let $P(v)$ be its orthogonal projection onto $H^\perp$. Then $v - P(v) \in H$. Thus

$$\begin{aligned} \inf \langle v, w \rangle &= \inf \langle v - Pv, w \rangle \\ &\leqslant -\alpha \|v - Pv\| \\ &= -\alpha \|v\| \sin \psi_v \\ &\leqslant -K \|v\| \psi_v, \end{aligned}$$

with a suitable real number $K > 0$. ∎

Corollary 2.2. *Let A be a non-empty bounded subset of $\mathbb{R}^N$ such that $0 \in \operatorname{con}(A)$ and $0 \notin \operatorname{con}(\tilde{A})$ for each $\tilde{A} \subsetneq A$.*

Then there exists a constant $K > 0$ such that

$$\underset{v \in \mathbb{R}^N}{\forall} \inf_{w \in A} \langle v, w \rangle \leqslant -K \|v\| \psi_v,$$

where ψ_v denotes the angle between v and $H^\perp := (\operatorname{span} A)^\perp$.

Proof. The assumptions of the corollary imply that A is a finite set, say

$$A = \{w^1, w^2, \ldots, w^k\}.$$

Since $0 \notin \operatorname{con}(\tilde{A})$ for each $\tilde{A} \subsetneq A$, there exist $\rho_1, \rho_2, \ldots, \rho_k > 0$ such that

$$\rho_1 + \rho_2 + \cdots + \rho_k = 1$$

and

$$\rho_1 w^1 + \rho_2 w^2 + \cdots + \rho_k w^k = 0.$$

Choose $v \in H \setminus \{0\}$. Then the last equation implies

$$\rho_1 \langle v, w^1 \rangle + \rho_2 \langle v, w^2 \rangle + \cdots + \rho_k \langle v, w^k \rangle = 0.$$

Since $v \in H$ and $\rho_i > 0$, at least one product $\langle v, w^j \rangle$ is different from zero. Consequently

$$\forall_{v \in H \setminus \{0\}} \inf_{w \in A} \langle v, w \rangle < 0.$$

Now apply Lemma 2.1. ∎

COROLLARY 2.3. *Let A be a non-empty bounded subset of $\mathbb{R}^N$ and $(A_\lambda)_{\lambda \in \Lambda}$ be a family of subsets of A such that $A = \bigcup A_\lambda$ and for each $\lambda \in \Lambda$*

$$0 \in \operatorname{con}(A_\lambda) \mathbin{\&} 0 \notin \operatorname{con}(\tilde{A}_\lambda) \qquad \text{if} \quad \tilde{A}_\lambda \subsetneq A_\lambda.$$

Then there exists a constant $K > 0$ such that

(a) $\forall_{v \in \mathbb{R}^N} \inf_{w \in A} \langle v, w \rangle \leqslant -K \|v\| \psi_v$,

(b) $\forall_{v \in \mathbb{R}^N} \inf_{w \in A} \langle v, w \rangle \leqslant -K \|v\| \sup_{\lambda \in \Lambda} \psi_v^\lambda$,

where ψ_v^λ denotes the angle between v and $H_\lambda^\perp := (\operatorname{span} A_\lambda)^\perp$.

Proof. By Corollary 2.2, there exists for each $\lambda \in \Lambda$ a constant $K_\lambda > 0$ such that

$$\forall_{v \in \mathbb{R}^N} \inf_{w \in A} \langle v, w \rangle \leqslant \inf_{w \in A_\lambda} \langle v, w \rangle \leqslant -K_\lambda \|v\| \psi_v^\lambda.$$

Consider $v \in H := \operatorname{span}(A)$, $v \neq 0$. Since $v \notin H^\perp$ and $H^\perp = \bigcap_{\lambda \in \Lambda} H_\lambda^\perp$ there exists $\lambda \in \Lambda$ such that $v \in H_\lambda^\perp$. Hence $\psi_v^\lambda > 0$. Consequently, we have

$$\forall_{\substack{v \in H \\ v \neq 0}} \inf_{w \in A} \langle v, w \rangle < 0.$$

Applying Lemma 2.1, we obtain (a).

Since $H^\perp \subset H_\lambda^\perp$, we have $\psi_v^\lambda \leqslant \psi_v$ for each $\lambda \in \Lambda$, and (b) follows immediately. ∎

3. Refined Kolmogorov Criteria

In the following we use the abbreviation

$$w := r_0 C - B,$$

where r_0 is a fixed element of V.

LEMMA 3.1. *Let ε be a primitive critical signature for $r_0 \in V$. Then*

$$0 \in \operatorname{con}\{\varepsilon(s)\, w(s) \in \mathbb{R}^N \mid s \in \mathrm{DOM}(\varepsilon)\}$$

and

$$0 \notin \operatorname{con}\{\varepsilon(s)\, w(s) \in \mathbb{R}^N \mid s \in F\}$$

for each $F \subsetneq \mathrm{DOM}(\varepsilon)$.

Proof. Let $\mathrm{DOM}(\varepsilon) =: \{s_1, s_2, ..., s_k\}$. Then there exist real numbers $\alpha_1, \alpha_2, ..., \alpha_k > 0$ such that

$$\sum_{i=1}^{k} \alpha_i \varepsilon(s_i)\, \varphi_j(s_i) = 0,$$

$j = 1, 2, ..., d$. Since each coordinate of w is an element of $\mathscr{L}(r_0)$, we have also

$$\sum_{i=1}^{k} \alpha_i \varepsilon(s_i)\, w(s_i) = 0,$$

which implies

$$0 \in \operatorname{con}\{\varepsilon(s)\, w(s) \in \mathbb{R}^N \mid s \in \mathrm{DOM}(\varepsilon)\}.$$

Suppose there exists a subset $F \subseteq \mathrm{DOM}(\varepsilon)$ (we can assume $F = \{s_1, s_2, ..., s_n\}$, $n < k$) and real numbers $\rho_1, \rho_2, ..., \rho_n > 0$ such that

$$\sum_{i=1}^{n} \rho_i \varepsilon(s_i)\, w(s_i) = 0.$$

Since

$$\mathscr{L}(r_0) = \{\langle w, v\rangle \in C(S) \mid v \in \mathbb{R}^N\},$$

we have

$$\sum_{i=1}^{n} \rho_i \varepsilon(s_i)\, h(s_i) = 0$$

for each $h \in \mathscr{L}(r_0)$. In particular, we have

$$\sum_{i=1}^{n} \rho_i \varepsilon(s_i)\, \varphi_j(s_i) = 0,$$

$j = 1, 2, \ldots, d$ or

$$\sum_{i=1}^{n} \rho_i G(\varepsilon(s_i), s_i) = 0,$$

i.e., the restriction of ε to the set F is critical. ∎

THEOREM 3.2 (Local Kolmogorov criterion). *Let (r_0, z_0) be a solution of* MPR(x). *Then there exists a constant $K > 0$ such that*

(a) $$\forall_{v \in \mathbb{R}^N} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v\rangle \leqslant -K\,\|v\|\, \varphi_v(\tilde{\varepsilon}_0);$$

(b) $$\forall_{v \in \mathbb{R}^N} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v\rangle \leqslant -K\,\|v\| \sup_{\varepsilon \in \Lambda_0} \varphi_v(\varepsilon).$$

Proof. The non-empty set

$$A := \{\varepsilon_0(s)\, w(s) \in \mathbb{R}^N \mid s \in S_0\}$$

is bounded, since it is contained in the compact set

$$\{\varepsilon_0(s)\, w(s) \in \mathbb{R}^N \mid s \in \mathrm{DOM}(\varepsilon_0)\}.$$

By definition of S_0 we have

$$A = \bigcup_{\varepsilon \in \Lambda_0} A_\varepsilon,$$

where

$$A_\varepsilon := \{\varepsilon_0(s)\, w(s) \in \mathbb{R}^N \mid s \in \mathrm{DOM}(\varepsilon)\}.$$

By Lemma 3.1 and by Corollary 2.3 there exists a constant $K > 0$ such that

(a) $$\forall_{v \in \mathbb{R}^N} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v\rangle \leqslant \inf_{s \in \mathrm{DOM}(\tilde{\varepsilon}_0)} \varepsilon_0(s)\langle w(s), v\rangle \leqslant -K\,\|v\|\, \varphi_v(\tilde{\varepsilon}_0);$$

(b) $$\forall_{v \in \mathbb{R}^N} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v\rangle \leqslant \inf_{s \in \mathrm{DOM}(\tilde{\varepsilon}_0)} \varepsilon_0(s)\langle w(s), v\rangle \leqslant -K\,\|v\| \sup_{\varepsilon \in \Lambda_0} \varphi_v(\varepsilon).$$ ∎

THEOREM 3.3 (Global Kolmogorov criterion). *Let* (r_0, z_0) *be a solution of* MPR(x). *Then there exists a constant* $K_1 > 0$ *such that*

$$\text{(a)} \quad \underset{r \in V}{\forall} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)(r_0(s) - r(s)) \leqslant -K_1 \varphi_v(\tilde{\varepsilon}_0);$$

$$\text{(b)} \quad \underset{r \in V}{\forall} \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)(r_0(s) - r(s) \leqslant -K_1 \sup_{\varepsilon \in \Lambda_0} \varphi_v(\varepsilon),$$

where $v \in U$ *is such that* $r = \langle B, v \rangle / \langle C, v \rangle$.

Proof. Let $\tilde{s} \in \mathrm{DOM}(\varepsilon_0)$ be such that

$$\varepsilon_0(\tilde{s})\langle w(\tilde{s}), v \rangle = \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v \rangle.$$

Then, by using Theorem 3.2 we have

$$\begin{aligned} &\min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)(r_0(s) - r(s)) \\ &\qquad = \min \frac{\varepsilon_0(s)\langle r_0(s)\, C(s) - B(s), v \rangle}{\langle C(s), v \rangle} \\ &\qquad \leqslant \frac{\varepsilon_0(\tilde{s})\langle r_0(\tilde{s})\, C(\tilde{s}) - B(\tilde{s}), v \rangle}{\langle C(\tilde{s}), v \rangle} \\ &\qquad \leqslant -\frac{K\, \|v\|\, \varphi_v(\tilde{\varepsilon}_0)}{\|C\|_\infty\, \|v\|} =: -K_1 \varphi_v(\tilde{\varepsilon}_0), \end{aligned}$$

which proves (a).

Since $V(\tilde{\varepsilon}_0) \subset V(\varepsilon)$ for each $\varepsilon \in \Lambda_0$, we have $\varphi_v(\tilde{\varepsilon}_0) \geqslant \varphi_v(\varepsilon)$, which implies (b). ∎

Remark. Instead of estimating $\langle C(s), v \rangle$ by $\|C\|_\infty \cdot \|v\|$ we could have used the sharper estimate $\langle C(s), v \rangle \leqslant \|C\|_\infty \cdot \|\bar{v}\|$, where $\bar{v} \in \mathbb{R}^N$ is defined by

$$\begin{aligned} \bar{v}_i &:= v_i \quad &&\text{if} \quad C_i \neq 0 \\ &:= 0 \quad &&\text{if} \quad C_i = 0, \end{aligned}$$

$i = 1, 2, \ldots, N$. This would imply also the sharper estimate

$$\underset{(v,z) \in Z_z}{\forall} \quad z \geqslant z_0 + \frac{K\, \|v\|}{\|\bar{v}\|}\, \varphi_v$$

in the sufficiency part of Theorem 4.1.

In the case of linear problems the refined Kolmogorov criterion can be stated in a more simplified way. Consider the particuar situation

$$B(s) := (g_1(s),\ g_2(s), \ldots, g_l(s), 0),$$
$$C(s) := (0, 0, \ldots, 0, 1),$$

where $g_1, g_2, \ldots, g_l$ are linearly independent functions of $C(S)$. Then for each $x \in C(S)$ we have the linear problem MPL(x).

Minimize $p(v, v) := z$

Subject to

$$\underset{(\eta,s) \in T}{\forall}\ \eta\left(\frac{\sum_{i=1}^{l} v_i g_i(s)}{v_{l+1}} - x(s)\right) \leqslant \gamma(\eta, s) z.$$

For any signature ε we introduce the linear subspaces

$$V_L(\varepsilon) := \left\{ b \in \mathbb{R}^l \,\middle|\, \underset{s \in \mathrm{DOM}(\varepsilon)}{\forall}\ \sum_{i=1}^{l} b_i g_i(s) = 0 \right\}$$

and

$$V_R(\varepsilon) := \left\{ v \in \mathbb{R}^{l+1} \,\middle|\, \underset{s \in \mathrm{DOM}(\varepsilon)}{\forall}\ \langle B(s), v \rangle = 0 \right\}.$$

Let $I\colon \mathbb{R}^l \to \mathbb{R}^{l+1}$ be the injection defined by

$$\underset{b \in \mathbb{R}^l}{\forall}\ I(b) := (b, 0).$$

Then we have

$$V_R(\varepsilon) = I(V_L(\varepsilon)) \oplus \mathbb{R} e_{l+1}. \tag{*}$$

Let $P_R\colon \mathbb{R}^{l+1} \to V_R(\varepsilon)$ and $P_L\colon \mathbb{R}^l \to V_L(\varepsilon)$ be the projections associated with the spaces $V_R(\varepsilon)$ and $V_L(\varepsilon)$, respectively. Then we have

$$P_R \circ I = I \circ P_L.$$

To prove this relation choose an element $b \in \mathbb{R}^l$. Then we have

$$\underset{u \in V_L(\varepsilon)}{\forall}\ \langle b - P_L b, u \rangle = 0,$$

which is equivalent to

$$\underset{v \in I(V_L(\varepsilon))}{\forall}\ \langle I(b) - I \circ P_L(b), v \rangle = 0.$$

By (*) we also have

$$\underset{v \in V_R(\varepsilon)}{\forall} \quad \langle I(b) - I \circ P_L(b), v \rangle = 0.$$

Hence $I \circ P_L(b)$ is the projection of $I(b)$ onto $V_R(\varepsilon)$, i.e., $P_R \circ I = I \circ P_L$.

THEOREM 3.4. (Refined linear Kolmogorov criterion). *Let* (g_0, z_0) *be a solution of* MPL(x). *Then there exists a real number* $K_2 > 0$ *such that*

$$\text{(a)} \quad \underset{g \in V}{\forall} \quad \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\, g(s) \leqslant -K_2 \, \|g\|_\infty \cdot \theta_g(\tilde{\varepsilon}_0)$$

$$\text{(b)} \quad \underset{g \in V}{\forall} \quad \min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\, g(s) \leqslant -K_2 \, \|g\|_\infty \cdot \sup_{\varepsilon \in \Lambda_0} \theta_g(\varepsilon),$$

where $\theta_g(\varepsilon)$ *denotes the angle between* $V_L(\varepsilon)$ *and* b, $g = \sum_{i=1}^{l} b_i g_i$.

Proof. We can assume $g_0 = 0$. Let $g = \sum_{i=1}^{l} b_i g_i$ be given. By using Theorem 3.2 with $v = I(b) + e_{l+1}$ we have for a suitable $K_3 > 0$

$$\begin{aligned}
&\min_{s \in \mathrm{DOM}(\varepsilon_0)} \varepsilon_0(s)\, g(s) \\
&\quad \leqslant -K_3 \, \|I(b) + e_{l+1}\| \sin \varphi_v(\tilde{\varepsilon}_0) \\
&\quad = -K_3 \, \|I(b) + e_{l+1} - P_R(I(b) + e_{l+1})\| \\
&\quad = -K_3 \, \|I(b) - P_R \circ I(b)\| \\
&\quad = -K_3 \, \|I(b) - I \circ P_L(b)\| \\
&\quad = -K_3 \, \|b - P_L(b)\| = -K_3 \, \|b\| \sin \theta_g(\tilde{\varepsilon}_0) \leqslant -K_2 \, \|g\|_\infty \, \theta_g(\tilde{\varepsilon}_0),
\end{aligned}$$

which proves (a).

Statement (b) follows from (a) by using the fact $\theta_g(\tilde{\varepsilon}_0) \geqslant \theta_g(\varepsilon)$ for each $\varepsilon \in \Lambda_0$. ∎

4. A NECESSARY AND SUFFICIENT CONDITION FOR STRONG UNIQUENESS

For each $r_0 = \langle B, v_0 \rangle / \langle C, v_0 \rangle$ in V the linear subspace

$$H_0 := \left\{ y \in \mathbb{R}^N \;\middle|\; \underset{s \in S}{\forall} \; \langle r_0(s)\, C(s) - B(s), y \rangle = 0 \right\}$$

has dimension $N - d$. In fact, define the linear mapping $F\colon \mathbb{R}^N \to C(S)$ by setting

$$\underset{v \in \mathbb{R}^N}{\forall} \; F(v) := \langle r_0 C - B, v \rangle.$$

Then we have $\mathrm{KER}(F) = H_0$ and $\mathrm{IM}(F) = \mathscr{L}(r_0)$, which proves $N = \dim H_0 + d$.

THEOREM 4.1. *Let (r_0, z_0) be a solution of* MPR(x). *Consider the following conditions*:

(a) *There exist points $s_i \in S_0$, $i = 1, 2,..., d$, such that the vectors*

$$r_0(s_i)\, C(s_i) - B(s_i) \in \mathbb{R}^N,$$

$i = 1, 2,..., d$, are linearly independent.

(b) *There exists a constant $K := K(x) > 0$ such that*

$$\underset{(v,z) \in Z_x}{\forall} \quad z \geqslant z_0 + K\varphi_v.$$

Then (a) $\Rightarrow$ (b). *Moreover, if $\gamma(\eta, s) > 0$ for all $(\eta, s) \in T$ then we also have* (b) $\Rightarrow$ (a).

Proof. (a) $\Rightarrow$ (b). We show that $H_0 = V(\tilde{\varepsilon}_0)$. The inclusion $H_0 \subset V(\tilde{\varepsilon}_0)$ is clear. On the other hand there exist signatures $\varepsilon_1, \varepsilon_2,..., \varepsilon_k$ in Λ_0 such that

$$\{s_1, s_2,..., s_d\} \subset \bigcup_{i=1}^{k} \mathrm{DOM}(\varepsilon_i).$$

The linear subspace

$$H^{\#} := \{v \in \mathbb{R}^N \mid \langle r_0(s_i)\, C(s_i) - B(s_i), v\rangle = 0, i = 1, 2,..., d\}$$

has dimension $N - d$ and contains $V(\tilde{\varepsilon}_0)$. Thus we have

$$H_0 \subset V(\tilde{\varepsilon}_0) \subset H^{\#}.$$

Since $\dim H_0 = N - d$, we have

$$H_0 = V(\tilde{\varepsilon}_0) = H^{\#}.$$

Consequently we have $\varphi_v = \varphi_v(\tilde{\varepsilon}_0)$ for each $v \in \mathbb{R}^N$.

Let (v, z) be in Z_x and let $r = \langle B, v\rangle / \langle C, v\rangle$. By theorem 3.3(a) there exist $K_1 > 0$ and a pair $(\varepsilon_0(s), s) \in M_0$ such that

$$\varepsilon_0(s)(r_0(s) - r(s)) \leqslant -K_1 \varphi_v(\tilde{\varepsilon}_0).$$

Then we have

$$\begin{aligned}\|\gamma\|_\infty(z-z_0) &\geqslant \gamma(\varepsilon_0(s), s)(z-z_0)\\ &\geqslant \varepsilon_0(s)(r(s)-x(s)) - \varepsilon_0(s)(r_0 - x(s))\\ &= -\varepsilon_0(s)(r_0(s)-r(s))\\ &\geqslant K_1\varphi_v(\tilde{\varepsilon}_0),\end{aligned}$$

which implies

$$z - z_0 \geqslant K\varphi_v$$

where $K := K_1/\|\gamma\|_\infty$.

(b) $\Rightarrow$ (a). Consider

$$S_1 := \operatorname{span}\{r_0(s)\, C(s) - B(s) \in \mathbb{R}^N \mid s \in S_0\},$$

let $d_1 := \dim S_1$ and assume by contradiction $d_1 < d$. Since $\dim S_1^\perp = N - d_1$, $\dim H_0^\perp = d$, and $d - d_1 > 0$, we have

$$\dim(S_1^\perp \cap H_0^\perp) \geqslant 1.$$

Now we claim that we can choose $v \in S_1^\perp \cap H_0^\perp$, $v \neq 0$, such that

$$\underset{(\varepsilon_0(s),s)\in M_0\backslash\Gamma_0}{\forall} \quad \varepsilon_0(s)\langle B(s) - r_0(s)\, C(s), v\rangle \leqslant 0.$$

If not, there exists for each $v \in S_1^\perp \cap H_0^\perp$, $v \neq 0$, a point $(\varepsilon_0(s), s) \in M_0\backslash\Gamma_0$ such that

$$\varepsilon_0(s)\langle y(s), v\rangle > 0,$$

where we have used the abbreviation

$$y(s) := B(s) - r_0(s)\, C(s).$$

Consequently, the convex hull of the linear functionals

$$x_s^*: v \mapsto \varepsilon_0(s)\langle y(s), v\rangle,$$

$(\varepsilon_0(s), s) \in M_0\backslash\Gamma_0$, defined on $H_0^\perp \cap S_1^\perp$ has a non-empty interior. If not, there exists $x^* \in (H_0^\perp \cap S_1^\perp)^*$ orthogonal to x_s^* for all $(\varepsilon_0(s), s) \in M_0\backslash\Gamma_0$. So, for some $v \in H_0^\perp \cap S_0^\perp$ we would have

$$\begin{aligned}\underset{(\varepsilon_0(s),s)\in M_0\backslash\Gamma_0}{\forall} \quad 0 = \langle x_s^*, x^*\rangle &= x_s^*(v)\\ &= \varepsilon_0(s)\langle y(s), v\rangle,\end{aligned}$$

which is impossible. Further we claim

$$0 \in \operatorname{con}\{x_s^* \in (H_0^\perp \cap S_1^\perp)^* \mid (\varepsilon_0(s), s) \in M_0 \backslash \Gamma_0\}.$$

If not, there exists an element $v \in (S_1^\perp \cap H_0^\perp) \backslash \{0\}$ such that for all elements

$$a \in \operatorname{con}\{x_s^* \in (S_1^\perp \cap H_0^\perp)^* \mid (\varepsilon_0(s), s) \in M_0 \backslash \Gamma_0\}$$

we have $\langle a, v \rangle \leqslant 0$, which implies

$$\varepsilon_0(s) \langle y(s), v \rangle \leqslant 0$$

for all $(\varepsilon_0(s), s) \in M_0 \backslash \Gamma_0$.

Consequently, there exist real numbers

$$\tau_1, \tau_2, \ldots, \tau_k > 0$$

and points

$$(\varepsilon_0(s_1), s_1), (\varepsilon_0(s_2), s_2), \ldots, (\varepsilon_0(s_k), s_k) \in M_0 \backslash \Gamma_0$$

such that $\tau_1 + \tau_2 + \cdots + \tau_k = 1$ and

$$\forall_{v \in S_1^\perp \cap H_0^\perp} \left\langle \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i)\, y(s_i), v \right\rangle = 0.$$

By assumption, there exist d_1 points $p_1, p_2, \ldots, p_{d_1}$ in S_0 such that the set of vectors

$$\{y(p_1), y(p_2), \ldots, y(p_{d_1})\}$$

is linearly independent. Choose a finite number signatures

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \Lambda_0$$

such that

$$\{p_1, p_2, \ldots, p_{d_1}\} \subset \bigcup_{i=1}^{n} \operatorname{DOM}(\varepsilon_i).$$

Denote the points in $\bigcup \operatorname{DOM}(\varepsilon_i)$ by $p_1, p_2, \ldots, p_m$. Then there exist real numbers $\rho_1, \rho_2, \ldots, \rho_m > 0$ such that

$$\sum_{i=1}^{m} \rho_i G(\varepsilon_0(p_i), p_i) = 0,$$

which implies

$$z := \sum_{i=1}^{m} \rho_i \varepsilon_0(p_i)\, y(p_i) = 0.$$

Choose a basis $v_1, v_2,..., v_{d_1}$ of S_1. Then the matrix

$$(\varepsilon_0(p_i)\langle y(p_i), v_j\rangle)_{\substack{j=1,2,...,d_1\\ i=1,2,...,m}}$$

has rank d_1, and consequently the linear system

$$\sum_{i=1}^{m} \lambda_i \varepsilon_0(p_i)\langle y(p_i), v_j\rangle = -\left\langle \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i) y(s_i), v_j \right\rangle,$$

$j = 1, 2,..., d_1$, has a solution

$$(\lambda_1, \lambda_2,..., \lambda_m) = \mathbb{R}^m.$$

With the aid of this solution define the element

$$\tilde{y} := \sum_{i=1}^{m} \lambda_i \varepsilon_0(p_i)\, y(p_i) + \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i)\, y(s_i).$$

Each element $v \in \mathbb{R}^N$ can be represented as $v = w_0 + w_1 + w_2$, where $w_0 \in H_0$, $w_1 \in H_0^\perp \cap S_1$, and $w_2 \in H_0^\perp \cap S_1^\perp$. Using this representation, an easy calculation shows

$$\forall_{v \in \mathbb{R}^N} \langle \tilde{y}, v \rangle = 0.$$

We can find $\tau \in \mathbb{R}$ such that all coefficients

$$\tilde{\rho}_i := \lambda_i + \tau \rho_i,$$

$i = 1, 2,..., m$, are positive and at least one is zero. Without loss of generality we can assume $\tilde{\rho}_i > 0$ for $i = 1, 2,..., m_1 < m$ and $\tilde{\rho}_i = 0$ for $i = m_1 + 1$, $m_1 + 2,..., m$. Thus we have

$$\tilde{y} + \tau z = \sum_{i=1}^{m_1} \tilde{\rho}_i \varepsilon_0(p_i)\, y(p_i) + \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i)\, y(s_i).$$

Of course, we also have

$$\forall_{v \in \mathbb{R}^N} \langle \tilde{y} + \tau z, v \rangle = 0.$$

Now assume ε_0 restricted to the set

$$\{p_1, p_2, ..., p_{m_1}\}$$

is critical. Then there exist real numbers

$$\hat{\rho}_1, \hat{\rho}_2, ..., \hat{\rho}_{m_1} \geqslant 0$$

such that $\hat{\rho}_1 + \hat{\rho}_2 + \cdots + \hat{\rho}_{m_1} = 1$ and

$$\tilde{z} := \sum_{i=1}^{m_1} \hat{\rho}_i \varepsilon_0(p_i)\, y(p_i) = 0.$$

We can find $\tilde{\tau} > 0$ such that all coefficients

$$\bar{\rho}_i := (\tilde{\rho}_i - \tilde{\tau}\hat{\rho}_i),$$

$i = 1, 2, ..., m_1$, are non-negative and at least one is zero. Without loss of generality we can assume $\bar{\rho}_i > 0$ for $i = 1, 2, ..., m_2 < m_1$ and $\bar{\rho}_i = 0$ for $i = m_2 + 1, m_2 + 2, ..., m_1$. Thus we have

$$\tilde{y} + \tau z - \tilde{\tau}\tilde{z} = \sum_{i=1}^{m_2} \bar{\rho}_i \varepsilon_0(p_i)\, y(p_i) + \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i)\, y(s_i)$$

which satisfies the relation

$$\forall_{v \in \mathbb{R}^N} \langle \tilde{y} + \tau z - \tilde{\tau}\tilde{z}, v \rangle = 0.$$

By repeating this process, if necessary, we can assume that the restriction of ε_0 to the set $\{p_1, p_2, ..., p_{m_2}\}$, $0 \leqslant m_2 < m$ is not critical.

The points $p_1, p_2, ..., p_{m_2}, s_1, s_2, ..., s_k$ satisfy the relation

$$\sum_{i=1}^{m_2} \bar{\rho}_i \varepsilon_0(p_i)\, y(p_i) + \sum_{i=1}^{k} \tau_i \varepsilon_0(s_i)\, y(s_i) = 0$$

with $\bar{\rho}_i > 0$ and $\tau_i > 0$. Then we also have

$$\sum_{i=1}^{m_2} \bar{\rho}_i G(\varepsilon_0(p_i), p_i) + \sum_{i=1}^{k} \tau_i G(\varepsilon_0(s_i), s_i) = 0,$$

i.e., the restriction of ε_0 to the set

$$\{p_1, p_2, ..., p_{m_2}\} \cup \{s_1, s_2, ..., s_k\}$$

is critical. Then there exists $\bar{\varepsilon} \in \Lambda_0$ such that

$$\mathrm{DOM}(\bar{\varepsilon}) \cap \{s_1, s_2, ..., s_k\} \neq \emptyset,$$

which contradicts the definition of Γ_0.

Thus, our first claim is proved, i.e., there exists $v \in H_0^\perp \cap S_1^\perp$, $v \neq 0$, such that

$$\underset{(\varepsilon_0(s), s) \in M_0 \backslash \Gamma_0}{\forall} \quad \varepsilon_0(s) \langle B(s) - r_0(s)\, C(s), v \rangle \leqslant 0.$$

Since $v \in S_1^\perp$, we also have

$$\underset{(\varepsilon_0(s), s) \in M_0}{\forall} \quad \varepsilon_0(s) \langle B(s) - r_0(s)\, C(s), v \rangle \leqslant 0.$$

Define a sequence of positive real numbers (τ_n) such that $\tau_n < 1$, $\tau_n \to 0$ for $n \to \infty$, and

$$v_n := (1 - \tau_n) v_0 + \tau_n v$$

belongs to U. Since $v_0 \in H_0$ and $v \in H_0^\perp$, we have

$$\begin{aligned} \sin \varphi_n &= \frac{\|v_n - Pv_n\|}{\|v_n\|} \\ &= \frac{\tau_n \|v\|}{\|v_n\|} \\ &= \frac{\tau_n \|v\|}{\|(1 - \tau_n)\, v_0 + \tau_n v\|} \\ &\geqslant K_0 \tau_n, \end{aligned}$$

with a suitable constant $K_0 > 0$, $\varphi_n := \varphi_{v_n}$, and where P denotes the projection associated with H_0.

For each $n \in \mathbb{N}$ we define a real number z_n and a point $(\eta_n, s_n) \in T$ such that

$$\begin{aligned} z_n &= \frac{\eta_n(r_n(s_n) - x(s_n))}{\gamma(\eta_n, s_n)} \\ &= \sup \left\{ \frac{\eta(r_n(s) - x(s))}{\gamma(\eta, s)} \in \mathbb{R} \;\middle|\; (\eta, s) \in T \right\}, \end{aligned}$$

where $r_n := \langle B, v_n \rangle / \langle C, v_n \rangle$. (We remark that the existence of such points $(\eta_n, s_n) \in T$ follows from the assumption $\gamma > 0$.)

There is an infinite subset $N_0 \subset \mathbb{N}$ such that either

$$\{(\eta_n, s_n) \in T \mid n \in N_0\}$$

consists of a single point, say $(\bar{\eta}, \bar{s})$, or, by compactness of T,

$$\{(\eta_n, s_n) \in T \mid n \in N_0\}$$

has an accumulation point $(\bar{\eta}, \bar{s})$ in T. By hypothesis, we have with a suitable constant $K_1 > 0$ and for all $n \in N_0$ the inequality

$$\begin{aligned}
0 < KK_1 \tau_n &\leqslant K\varphi_n \leqslant z_n - z_0 \\
&\leqslant \frac{\eta_n(r_n(s_n) - x(s_n))}{\gamma(\eta_n, s_n)} - \frac{\eta_n(r_0(s_n) - x(s_n))}{\gamma(\eta_n, s_n)} \\
&= \frac{\tau_n \eta_n \langle B(s_n) - r_0(s_n)\, C(s_n), v\rangle}{\gamma(\eta_n, s_n)\langle C(s_n), v_n\rangle}
\end{aligned}$$

which implies

$$0 < KK_1 \leqslant \frac{\eta_n \langle B(s_n) - r_0(s_n)\, C(s_n), v\rangle}{\gamma(\eta_n, s_n)\langle C(s_n), v_n\rangle}.$$

By continuity and since $(\bar{\eta}, \bar{s}) \in M_0$ we have

$$\begin{aligned}
0 < KK_1 &\leqslant \frac{\bar{\eta}\langle B(\bar{s}) - r_0(\bar{s})\, C(\bar{s}), v\rangle}{\gamma(\bar{\eta}, \bar{s})\langle C(\bar{s}), v_0\rangle} \\
&\leqslant 0. \quad \blacksquare
\end{aligned}$$

Following the remark after Theorem 3.3 we introduce the set

$$Z_x^{\#} := \{(v, z) \in Z_x \mid \|\bar{v}\| = 1\}.$$

From Theorem 4.1 we can derive the following generalization of a result of Cheney and Loeb [6]:

THEOREM 4.2. *Let* (r_0, z_0) *be a solution of* MPR(x). *Then condition* (b) *of Theorem* 4.1 *is equivalent to the condition*

(c) *There exists a constant* $K_1 := K_1(x) > 0$ *such that*

$$\underset{(v,z) \in Z_x^{\#}}{\forall} \quad z \geqslant z_0 + K_1 \cdot \operatorname{dist}(v, H_0);$$

consequently condition (a) *of Theorem* 4.1 *implies* (c), *and if* $\gamma(\eta, s) > 0$ *for all* $(\eta, s) \in T$ *then we also have* (c) $\Rightarrow$ (a).

Proof. (b) ⇒ (c). Using the remark after Theorem 3.3 we have the estimate

$$\begin{aligned}\forall_{(v,z)\in Z_x^{\#}} \quad z &\geqslant z_0 + K\,\|v\|\,\varphi_v \\ &\geqslant z_0 + K_1\,\|v\|\sin\varphi_v \\ &= z_0 + K_1\,\|v - Pv\| \\ &= z_0 + K_1\,\mathrm{dist}(v, H_0).\end{aligned}$$

(c) ⇒ (b). Choose $(v, z) \in Z_x$. Then we have

$$\begin{aligned} z &\geqslant z_0 + K_1\,\mathrm{dist}\left(\frac{v}{\|\bar{v}\|}, H_0\right) \\ &= z_0 + K_1 \cdot \frac{\|v - Pv\|}{\|\bar{v}\|} \\ &\geqslant z_0 + K_1 \cdot \frac{\|v - Pv\|}{\|v\|} \\ &\geqslant z_0 + K\varphi_v. \quad \blacksquare\end{aligned}$$

5. Strong Unicity in the Normal Case

An element $r_0 \in V$ is said to be normal iff $\dim \mathscr{L}(r_0) = N - 1$. A function x in L is also said to be normal iff there exists a solution (r_0, z_0) of $\mathrm{MPR}(x)$ such that r_0 is normal. For each $r \in V$ we can find $v \in U$ such that

$$r = \frac{\langle B, v\rangle}{\langle C, v\rangle} \qquad \text{and} \qquad \langle C(s_0), v\rangle = 1$$

for some s_0. We denote by Z_x^* the set

$$\{(v, z) \in Z_x \mid \langle C(s_0), v\rangle = 1\}.$$

If r_0 is normal, then $\dim H_0 = 1$. This implies that there exists a unique $v_0 \in H_0$ such that

$$r_0 = \frac{\langle B, v_0\rangle}{\langle C, v_0\rangle} \qquad \text{and} \qquad \langle C(s_0), v_0\rangle = 1.$$

We introduce the linear subspace

$$R_{N-1} := \{w \in \mathbb{R}^N \mid \langle C(s_0), w\rangle = 0\}$$

and we denote by $P\colon \mathbb{R}^N \to H_0$ the orthogonal projection associated with H_0.

LEMMA 5.1. *Let x be a normal point and let $(v_0, z_0) \in Z_x^*$ be a solution of* MPR(x). *Suppose there exists a constant $K > 0$ such that*

$$\forall_{(v,z) \in Z_x} \quad z - z_0 \geqslant K \sin \varphi_v.$$

Then there exists a constant $K_1 > 0$ such that

$$\forall_{(v,z) \in Z_x^*} \quad z - z_0 \geqslant \frac{K_1 \|v - v_0\|}{\|v\|}.$$

Proof. Let $(v, z) \in Z_x^*$. Then $v - v_0$ is in R_{N-1}. Since $H_0 \cap R_{N-1} = \{0\}$, the restriction of P to R_{N-1} has norm $0 < \mu < 1$. Then we have

$$\begin{aligned} z - z_0 &\geqslant K \sin \varphi_v \\ &= \frac{K \|v - Pv\|}{\|v\|} \\ &= \frac{K \|v - v_0 - P(v - v_0)\|}{\|v\|} \\ &\geqslant \frac{K(1-\mu) \|v - v_0\|}{\|v\|} \\ &=: \frac{K_1 \|v - v_0\|}{\|v\|} \quad \blacksquare \end{aligned}$$

THEOREM 5.2. *Let x be a normal point and let (r_0, z_0) be a solution of* MPR(x). *Then the following statements are equivalent*:

(a) *There exists a constant $K_a > 0$ such that*

$$\forall_{(v,z) \in Z_x} \quad z \geqslant z_0 + K_a \varphi_v.$$

(b) *There exists a constant $K_b > 0$ such that*

$$\forall_{(r,z) \in V_x} \quad z \geqslant z_0 + K_b \|r - r_0\|_\infty.$$

(c) *For each $\rho > 0$ there exists a constant $K_\rho > 0$ such that*

$$\forall_{\substack{(v,z) \in Z_x^* \\ \|v\| \leqslant \rho}} \quad z \geqslant z_0 + K_\rho \|v - v_0\|.$$

Proof. (a) ⇒ (c). By Lemma 5.1 there exists a constant $K>0$ such that

$$\underset{(v,z)\in Z_x^*}{\forall} \quad z \geqslant z_0 + \frac{K\,\|v-v_0\|}{\|v\|},$$

which implies (c).

(c) ⇒ (b). Assume by contradiction:

$$\underset{n\in\mathbb{N}}{\forall} \quad \underset{(v_n,z_n)\in Z_x}{\exists} \quad z_n - z_0 < \frac{1}{n}\,\|r_n - r_0\|_\infty,$$

where $r_n = \langle B, v_n\rangle / \langle C, v_n\rangle$. We can assume that $(v_n, z_n)\in Z_x^*$ and $v_n/\|v_n\| \to \bar{v}$.

We claim that $\|r_n - r_0\|_\infty$ is bounded. In fact, since

$$\underset{(\eta,s)\in T}{\forall} \quad \|\gamma\|_\infty z \geqslant \eta(r(s) - x(s))$$

it follows that

$$z \geqslant \|r-x\|_\infty / \|\gamma\|_\infty. \tag{*}$$

We have

$$0 < \frac{z_n - z_0}{\|r_n - r_0\|_\infty} < \frac{1}{n},$$

which implies

$$0 < \frac{z_n - z_0}{\|x - r_n\|_\infty + \|x - r_0\|_\infty} < \frac{1}{n},$$

consequently

$$0 < \frac{z_n - z_0}{\|\gamma\|_\infty (z_n + z_0)} < \frac{1}{n},$$

which implies that (z_n) is bounded and, by (*), that $\|r_n - r_0\|_\infty$ is also bounded. It follows that $z_n \to z_0$.

We claim that also the sequence $(\|v_n\|)$ is bounded. If not, then we have

$$\langle C(s_0), \bar{v}\rangle = 0.$$

Choose a $\tau > 0$ such that $v_0 + \tau\bar{v} \in U$ and introduce the abbreviation $w_n := v_n/\|v_n\|$. Then we have for each $n\in N$ and $(\eta, s)\in T$:

$$\eta\left(\frac{\langle B(s), v_0+\tau w_n\rangle}{\langle C(s), v_0+\tau w_n\rangle}-x(s)\right)$$
$$=\frac{\eta\langle C(s), v_0\rangle}{\langle C(s), v_0+\tau w_n\rangle}\left[\frac{\langle B(s), v_0\rangle}{\langle C(s), v_0\rangle}-x(s)\right]$$
$$+\frac{\eta\tau\langle C(s), w_n\rangle}{\langle C(s), v_0+\tau w_n\rangle}\left[\frac{\langle B(s), w_n\rangle}{\langle C(s), w_n\rangle}-x(s)\right]$$
$$\leqslant\frac{\langle C(s), z_0 v_0+z_n\tau w_n\rangle}{\langle C(s), v_0+\tau w_n\rangle}\gamma(\eta, s).$$

For $n\to\infty$ we obtain

$$\eta\left(\frac{\langle B(s), v_0+\tau\bar{v}\rangle}{\langle C(s), v_0+\tau\bar{v}\rangle}-x(s)\right)\leqslant z_0\gamma(\eta, s).$$

Consequently $(v_0+\tau\bar{v}, z_0)$ is also a solution of $\mathrm{MPR}(x)$, which belongs to Z_x^*. From (c) we conclude

$$v_0+\tau\bar{v}=v_0,$$

which leads to $\bar{v}=0$, contradicting $\|\bar{v}\|=1$. Consequently, the sequence $(\|v_n\|)$ is bounded.

By hypothesis there exists a suitable constant $K>0$ such that

$$\underset{n\in\mathbb{N}}{\forall}\ \frac{1}{n}\|r_n-r_0\|_\infty>z_n-z_0\geqslant K\|v_n-v_0\|,$$

which implies $v_n\to v_0$. Thus, there exists an $\rho>0$ and an $n_0\in\mathbb{N}$ such that

$$\underset{n\geqslant n_0}{\forall}\ \underset{s\in S}{\forall}\ \langle C(s), v_n\rangle\geqslant\rho>0.$$

So we have

$$\|r_n-r_0\|_\infty\leqslant\frac{\|\langle B-r_0C, v_n-v_0\rangle\|_\infty}{\rho}$$
$$\leqslant\frac{\|B-r_0C\|_\infty}{\rho}\|v_n-v_0\|$$
$$\leqslant\frac{\|B-r_0C\|_\infty}{\rho\cdot K\cdot\eta}\|r_n-r_0\|_\infty,$$

which implies

$$1 \leqslant \frac{1}{n} \cdot \frac{\|B - r_0 C\|_\infty}{\rho K}.$$

For $n \to \infty$ we obtain $1 \leqslant 0$, which is impossible.

(b) $\Rightarrow$ (a). Since for all $w \in R_{N-1} \backslash \{0\}$ we have

$$\|\langle B - r_0 C, w \rangle\|_\infty > 0,$$

there exists an $\alpha > 0$ such that

$$\underset{w \in R_{N-1}}{\forall} \quad \|\langle B - r_0 C, w \rangle\|_\infty \geqslant \alpha \, \|w\|.$$

Let $(v, z) \in Z_x$ and $r := \langle B, v \rangle / \langle C, v \rangle$. Then we have

$$\begin{aligned}
\|r - r_0\|_\infty &\geqslant \frac{\|\langle B - r_0 C, v - v_0 \rangle\|_\infty}{\|C\|_\infty \cdot \|v\|} \\
&\geqslant \frac{\alpha \, \|v - v_0\|}{\|C\|_\infty \, \|v\|} \\
&= \frac{\alpha}{2 \, \|C\|_\infty} \cdot \frac{\|v - v_0\| + \|v - v_0\|}{\|v\|} \\
&\geqslant \frac{\alpha}{2 \, \|C\|_\infty} \cdot \frac{\|v - v_0 - P(v - v_0)\|}{\|v\|} \\
&= \frac{\alpha}{2 \, \|C\|_\infty} \cdot \frac{\|v - Pv\|}{\|v\|} \\
&= \frac{\alpha}{2 \, \|C\|_\infty} \sin \varphi_v \geqslant K \varphi_v,
\end{aligned}$$

where $K > 0$ is a suitable constant. The last inequality and (b) imply

$$z \geqslant z_0 + K_b \, \|r - r_0\|_\infty \geqslant z_0 + K_a \varphi_v,$$

where $K_a := K \cdot K_b$. ∎

THEOREM 5.3. *Let x be a normal point and let (r_0, z_0) be a solution of* MPR(x). *Consider the following conditions*:

(a) *There exist points* $s_i \in S_0$, $i = 1, 2, ..., N-1$, *such that the vectors*

$$r_0(s_i)\, C(s_i) - B(s_i) \in \mathbb{R}^N,$$

$i = 1, 2, ..., N-1$, *are linearly independent.*

(b) *There exists a constant* $K := K(x) > 0$ *such that*

$$\forall_{(r,z) \in V_x} \quad z \geqslant z + K \, \|r - r_0\|_\infty.$$

Then (a) $\Rightarrow$ (b). *Moreover, if* $\gamma(\eta, s) > 0$ *for all* $(\eta, s) \in T$ *then we also have* (b) $\Rightarrow$ (a).

Proof. The theorem follows from Theorems 4.1 and 5.2. ∎

It is clear that we have a similar result for the local strong uniqueness in the parameter space using condition (c) of Theorem 5.2.

In the linear case (compare Section 3) we have $\mathscr{L}(r) = V$ for all $r \in V$. So the condition (a) of Theorem 5.3 reads:

There exist points $s_i \in S_0$, $i = 1, 2, ..., l := \dim V$ such that the vectors

$$(g_1(s_i), g_2(s_2), ..., g_l(s_i)) \in \mathbb{R}^l$$

$i = 1, 2, ..., l$ are linearly independent.

6. Some Remarks

In Theorems 3.2 and 3.3 the signature $\tilde{\varepsilon}_0$ cannot be replaced by ε_0 as the following example shows.

EXAMPLE 6.1. Let $S = [-1, 1]$, $\gamma(\eta, s) = 1$, and $V := \text{span}(g)$, where $g(s) = s$ for $s \in S$. Define a function $x \in C(S)$ by

$$\begin{aligned} x(s) &:= 1 && \text{if } \; 0 < s \leqslant 1 \\ &:= 1 - s^2 && \text{if } \; -1 \leqslant s \leqslant 0. \end{aligned}$$

Then the function $g_0(s) = 0$ defines a solution of the minimization problem MPR(x). We have

$$\begin{aligned} M_0 &= \{(-1, s) \in T \mid s \in [0, 1]\}; \\ \Gamma_0 &= \{(-1, 0)\}, \\ H_0 &= V(\varepsilon_0) = \{(0, v_2) \in \mathbb{R}^2 \mid v_2 \in \mathbb{R}\}; \end{aligned}$$

and $V(\tilde{\varepsilon}_0) = \mathbb{R}^2$.

If the statement (a) of Theorem 3.3 would be true for ε_0 instead of $\tilde{\varepsilon}_0$, then for $r(s)=s$ we would have

$$\begin{aligned} 0 = \min_{s \in [0,1]} s = \min_{s \in [0,1]} -1(0-s) \\ \leqslant -K_1 \varphi_v(\varepsilon_0) < 0. \end{aligned}$$

This example also shows that "φ^2-strong uniqueness" does not imply strong uniqueness. With the abbreviation $\alpha := v_1/v_2$ we have

$$\begin{aligned} \|x-\alpha g\|_\infty - \|x\|_\infty &= \frac{\alpha^2}{4} \quad \text{if} \quad \alpha \in [0,2] \\ &= \alpha - 1 \quad \text{if} \quad \alpha \geqslant 2 \\ &= -\alpha \quad \text{if} \quad \alpha \leqslant 0. \end{aligned}$$

Since we have

$$\sin \varphi_v = \sqrt{\frac{\alpha^2}{1+\alpha^2}},$$

we can find a constant $K>0$ such that

$$\left\| x - \frac{v_1}{v_2} g \right\|_\infty \geqslant K \varphi_v^2,$$

hence $g_0 = 0$ is a "φ^2-strongly unique" solution of $\mathrm{MPR}(x)$.

But there does not exist a constant $K_0 > 0$ such that

$$\left\| x - \frac{v_1}{v_2} g \right\|_\infty - \|x\|_\infty \geqslant K_0 \varphi_v.$$

Otherwise we would have

$$\|x-\alpha g\|_\infty - \|x\|_\infty = \frac{\alpha^2}{4} \geqslant K \sin \varphi = \frac{K\,|\alpha|}{\sqrt{1+\alpha^2}},$$

for all $\alpha \in [0,2]$. This implies

$$\frac{\sqrt{1+\alpha^2} \cdot |\alpha|}{4} \geqslant K$$

for all $\alpha \in [0,2]$, which is impossible. Hence g_0 is not a strongly unique solution of $\mathrm{MPR}(x)$. Of course, we could also have derived this result from Theorem 4.1. ∎

The next example shows that the condition $\gamma(\eta, s) > 0$ is necessary for proving the implication (b) $\Rightarrow$ (a) of Theorem 4.1.

EXAMPLE 6.2. Let $S = [0, 1]$, $\gamma(\eta, s) = (1 - \eta)/2$, and $V := \operatorname{span}(g)$, where $g(s) = s$ for each $s \in S$. Define a function $x \in C(S)$ by $x(s) = s^2$ for each $s \in S$.

Then $(0, 1)$ is a solution of MPR(x). We have

$$M_0 = \{(1, 0), (-1, 1)\}$$

and

$$\Gamma_0 = \{(1, 0)\}.$$

So condition (a) of Theorem 4.1 is not satisfied.

Since each feasible point (v, z) satisfies the inequality $\alpha := v_1/v_2 \leqslant 0$, we have

$$\begin{aligned} z - z_0 &= \|x - \alpha g\|_\infty - \|x\|_\infty \\ &= 1 - \alpha - 1 \\ &= -\alpha = |\alpha| \; \|g\|, \end{aligned}$$

i.e., $(0, 1)$ is a strongly unique solution of MPR(x).

In the linear case we can replace the condition $\gamma(\eta, s) > 0$ in the implication (b) $\Rightarrow$ (a) of Theorem 4.1 by another one. Define the sets

$$\begin{aligned} S^+ &:= \{s \in S \mid \gamma(1, s) = 0\}, \\ S^- &:= \{s \in S \mid \gamma(-1, s) = 0\}, \end{aligned}$$

$T^+ := \{1\} \times S^+$, and $T^- := \{-1\} \times S^-$. Then we have:

THEOREM 6.3. *Assume that there exists a function $\tilde{g} \in V$ such that $\tilde{g}(s) > 0$ on S^+ and $\tilde{g}(s) < 0$ on S^-. Let (g_0, z_0) be a solution of* MPL(x).
If there exists a constant $K > 0$ such that

$$\forall_{(g,z) \in V_x} \quad z - z_0 \geqslant K \|g - g_0\|_\infty,$$

then the condition (a) *of Theorem* 4.1 *is fulfilled.*

Proof. There exists an open set W containing $T^+ \cup T^-$ such that $\eta\tilde{g}(s) > 0$ for each $(\eta, s) \in W$. Let $\beta > 0$ be such that $\gamma(\eta, s) \geqslant \beta > 0$ for all

(η, s) in the compact set $T\backslash W$ and choose $\alpha > 0$ so small that $\beta > \alpha \|\tilde{g}\|_\infty$. Then

$$\bar{\gamma}(\eta, s) := \gamma(\eta, s) + \alpha\eta\tilde{g}(s)$$

is positive in T.

Now we consider the transformed minimization problem TMPL(x).

$$\text{Minimize } p(g, z) := z$$

$$\text{subject to}$$

$$\underset{(\eta,s)\in \Gamma}{\forall} \quad \eta(g(s) - x(s)) \leqslant \bar{\gamma}(\eta, s)z.$$

Then (g, z) is a feasible point of MPL(x) iff $(g + \alpha z\tilde{g}, z)$ is a feasible point of TMPL(x). This implies that (g, z) is a solution of MPL(x) iff $(g + \alpha z\tilde{g}, z)$ is a solution of TMPL(x). To prove the theorem, it suffices to prove that

$$(\bar{g}_0, z_0) := (g_0 + \alpha z_0 \tilde{g}, z_0)$$

is a strongly unique solution of TMPL(x).

Let $(\bar{g}, z)$ be a feasible point of TMPL(x), where $\bar{g} = g + \alpha z\tilde{g}$ with $(y, z) \in V_x$. Then we have

$$\begin{aligned}\|\bar{g} - \bar{g}_0\|_\infty &\leqslant \|g - g_0\|_\infty + (z - z_0)\,\|\alpha\tilde{g}\|_\infty \\ &\leqslant K(z - z_0) + (z - z_0)\,\|\alpha\tilde{g}\|_\infty \\ &=: K_0(z - z_0). \quad \blacksquare\end{aligned}$$

For the linear one-sided cases, i.e., $\gamma(\eta, s) = (1 + \eta)/2$ (resp. $\gamma(\eta, s) = (1 - \eta)/2$), we have $S^- = S$ and $S^+ = \varnothing$ (resp. $S^+ = S$ and $S^- = \varnothing$). Then we have the following:

COROLLARY 6.4. *Assume there exists a positive function in V. Then (g_0, z_0) is a strongly unique solution of* MPL(x) *iff condition* (a) *of Theorem* 4.1 *is fulfilled.* $\blacksquare$

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